SPECIAL CURVES AND POSTCRITICALLY-FINITE POLYNOMIALS

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1. Introduction

For each integer $d \geq 2$, let MP_d^{cm} denote the moduli space of critically-marked, complex polynomials of degree d.[1] We are interested in the postcritically-finite (PCF) polynomials within MP_d^{cm}, i.e., those polynomials whose critical points all have finite orbit under iteration. Such maps play a fundamental role in the theory of polynomial dynamics. The PCF polynomials form a countable and Zariski-dense subset of MP_d^{cm}; see Proposition 2.7 below. Our ultimate goal is to characterize algebraic subvarieties of MP_d^{cm} containing a Zariski-dense subset of PCF maps. In this paper, we make some concrete steps in this direction, focusing on certain kinds of algebraic curves in MP_d^{cm}. We also offer a conjecture for the general setting of subvarieties in the space of rational functions.

1.1. **Statement of main results.** To illustrate the idea, consider the following family of algebraic curves (introduced by Milnor in [Mi1]) in the space of critically-marked cubic polynomials:

$$\mathrm{Per}_1(\lambda) = \{f \in \mathrm{MP}_3^{cm} : f \text{ has a fixed point with multiplier } \lambda\}$$

for each $\lambda \in \mathbb{C}$. (Recall that the multiplier of a fixed point is simply the derivative of f at the fixed point.)

Theorem 1.1. *The curve* $\mathrm{Per}_1(\lambda)$ *contains infinitely many postcritically-finite cubic polynomials if and only if* $\lambda = 0$.

The idea of the proof is as follows. For $\lambda = 0$, one critical point is fixed for all $f \in \mathrm{Per}_1(0)$, so there is exactly one "active" critical point along each irreducible component of $\mathrm{Per}_1(0)$. By a classical complex dynamics argument, the active critical point

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[1]The moduli space MP_d^{cm} is the space of complex polynomials of degree d modulo conjugacy by conformal automorphisms of $\mathbb{C}$. It is a finite quotient of $\mathcal{P}_d^{cm} \simeq \mathbb{C}^{d-1}$, the space of critically-marked, monic and centered polynomials. Indeed, $\mathcal{P}_d^{cm}$ may be parameterized by tuples $(c_1, \ldots, c_{d-1}, b) \in \mathbb{C}^d$ such that $c_1 + \cdots + c_{d-1} = 0$. The associated polynomial is given by $f(z) = d \cdot \int_0^z \prod_i (\zeta - c_i)\, d\zeta + b$, with critical points at $\{c_1, \ldots, c_{d-1}\}$ and $b = f(0)$. Conjugating f by $z \mapsto \omega z$ where $\omega^{d-1} = 1$ induces an action of the cyclic group $\mathbb{Z}/(d-1)\mathbb{Z}$ on $\mathcal{P}_d^{cm}$ (coordinatewise multiplication by ω), and the moduli space MP_d^{cm} is the quotient of $\mathcal{P}_d^{cm}$ under this action.

must have finite forward orbit for a dense set of parameters in the bifurcation locus, so there are infinitely many PCF polynomials $f \in \mathrm{Per}_1(0)$. For the converse direction, assume there are infinitely many postcritically-finite maps in $\mathrm{Per}_1(\lambda)$. Then $\lambda \in \overline{\mathbb{Q}}$, and we apply an arithmetic equidistribution theorem (Theorem 3.1) to conclude that these PCF maps are equidistributed with respect to the bifurcation measure of each bifurcating critical point. In particular, if $\lambda \neq 0$, then the two critical points define the same bifurcation measure along $\mathrm{Per}_1(\lambda)$. But the two critical points are dynamically independent and must define distinct bifurcation measures, so we conclude that $\lambda = 0$; see §4.3 for details.

In general, we expect that an algebraic subvariety V in MP_d^{cm} contains a Zariski dense subset of PCF maps if and only if V is cut out by *critical orbit relations*. Unfortunately, pinning down a precise notion of "critical orbit relation" is a bit delicate, as we need to take into account the presence of nontrivial symmetries. In the next result, we provide a precise formulation for polynomially-parameterized curves in the space $\mathcal{P}_d^{cm}$, a branched cover of MP_d^{cm}, consisting of all monic and centered polynomials with marked critical points. We emphasize the equivalence of statements (1) and (4) in Theorem 1.2 below.

In order to state the result, we first need the following definitions. A *marked point* along a subvariety $V \subset \mathcal{P}_d^{cm}$ is a meromorphic function $a : V \to \mathbb{P}^1$; the marked point a is said to be *active* if it is not persistently preperiodic along V. When a parameterizes a critical point of $f \in V$, then activity means that the critical point is moving in and out of the Julia sets of $f \in V$ (via [MSS], [Mc1, Lemma 2.1], [DF, Theorem 2.5]). It follows that the *bifurcation measure* of the critical point on V is nonzero. See §2 for further details.

Theorem 1.2. *Let*

$$f_t = (c_1(t), \ldots, c_{d-1}(t), b(t)) \in \mathcal{P}_d^{cm}$$

be a family of polynomials, defined for $t \in \mathbb{C}$, where each coordinate function lies in $\mathbb{C}[t]$. The following are equivalent:

(1) *f_t is postcritically finite for infinitely many parameters t;*
(2) *for every pair of active critical points c_i and c_j, the normalized bifurcation measures are equal;*
(3) *the connectedness locus for $\{f_t\}$ is equal to*

$$\mathrm{M}_i = \left\{ t : \sup_n |f_t^n(c_i(t))| < \infty \right\}$$

for any choice of active critical point c_i;

(4) *for every pair of active critical points c_i and c_j, there exist a polynomial $h_t(z) \in \mathbb{C}[t, z]$ and integers $k > 0$, $n, m \geq 0$, such that*

$$h_t \circ f_t^k = f_t^k \circ h_t \qquad \textit{and} \qquad f_t^n(c_j(t)) = h_t(f_t^m(c_i(t)))$$

for all t.

In plain English, the equivalence of (1) and (4) means that there is a Zariski-dense set of parameters $t \in \mathbb{C}$ for which f_t is PCF *if and only if* there is exactly one active critical orbit up to symmetries (the h term). In particular, the critical point c_i has finite orbit for f_t if and only if c_j has finite orbit for f_t. If $\deg_z h = 1$, then h_t must be a symmetry of the Julia set of f_t; these were classified in [Be1]. If $\deg_z h > 1$, then h_t must share an iterate with f_t for all t [Ri2]; it follows that condition (4) is symmetric in i and j. In §1.3, we provide examples of polynomial families f_t satisfiying the conditions of Theorem 1.2, and we illustrate how we can use Theorem 1.2 to conclude that there are only finitely many postcritically-finite maps in certain explicit families.

Theorem 1.2 is a special case of the following result which concerns marked (but not necessarily critical) points which are simultaneously preperiodic.

Theorem 1.3. *Let f_t be a 1-parameter family of polynomials of degree $d \geq 2$, parameterized as*

$$f_t(z) = z^d + b_2(t)z^{d-2} + \cdots + b_d(t)$$

with $b_j(t) \in \mathbb{C}[t]$ for each j. Let $a_1(t), a_2(t) \in \mathbb{C}[t]$ be a pair of active marked points, and define

$$S_i := \{t \in \mathbb{C} : a_i(t) \text{ is preperiodic for } f_t\}.$$

The following are equivalent:

(1) $|S_1 \cap S_2| = \infty$
(2) $S_1 = S_2$
(3) *there exist a polynomial $h \in \mathbb{C}[t, z]$ and integers $k > 0$, $n, m \geq 0$ such that*

$$h_t \circ f_t^k = f_t^k \circ h_t \qquad \text{and} \qquad f_t^n(a_1(t)) = h_t f_t^m(a_2(t))$$

for all t.

Theorem 1.3 is an extension of the results [BD, Theorem 1.1] and [GHT1, Theorem 2.3], where stronger hypotheses guaranteed that the symmetries $\{h_t\}$ must be trivial. The new article [GHT2] is closely related, showing that (1) $\iff$ (2) for certain families of rational functions.

The idea behind our proof of Theorem 1.3 is as follows. If we assume condition (3), then (2) follows immediately and (1) follows from Montel's theorem (combined with Proposition 2.1), showing that an active point must have finite orbit at infinitely many parameters t. For the implication (1) $\implies$ (3), we begin by applying an arithmetic equidistribution theorem (Theorem 3.1) that implies an "almost (2)" statement: S_1 and S_2 can differ by at most finitely many elements. This step, which uses Berkovich analytic spaces in a crucial way, appeared in [GHT1] and we refer there for details.

To complete the proof that (1) implies (3), we use classical techniques from complex analysis to, first, deduce an analytic relation between the orbits of a_1 and a_2 and, then, promote this to an invariant algebraic relation. Finally, via recent results

of Medvedev-Scanlon [MS], employing methods of Ritt [Ri1] to classify invariant subvarieties for a certain class of polynomial dynamical systems, we may simplify the form of our algebraic relation to the statement of condition (3).

Theorem 1.1 is not a special case of Theorems 1.2 and 1.3, because the rational curves $\mathrm{Per}_1(\lambda)$ in $\mathcal{P}_3^{cm}$ are not parameterized by polynomials.

1.2. **Motivation from results in arithmetic geometry.** In arithmetic geometry, there are numerous results which fit into the following paradigm. One is given a complex algebraic variety X and a countable Zariski dense collection of "special" algebraic points on X. The question then arises which algebraic subvarieties of X can contain a Zariski dense set of special points. Usually one knows a family of "special subvarieties" of X which do contain a Zariski dense set of special points, and the problem is to determine whether an arbitrary subvariety of X containing a Zariski dense set of special points must itself be special.

The canonical example of this paradigm is the "Manin-Mumford conjecture", first established by Raynaud [Ra1, Ra2]. If X is an abelian variety then the torsion points of X are countable and Zariski dense, and if Y is a torsion subvariety of X (meaning a translate of an abelian subvariety by a torsion point) then Y contains a dense set of torsion points. Conversely, Raynaud's theorem asserts that if an algebraic subvariety Y of X contains a Zariski dense set of torsion points, then Y must be a torsion subvariety. An analogous result when X is an algebraic torus (so that torsion points are algebraic points of X whose coordinates are all roots of unity) was proved by Laurent, and extended to semiabelian varieties by Hindry [La], [Hi].

A more recent (and in general still conjectural) illustration of the special point and special subvariety formalism is the "André-Oort conjecture"; see e.g. [An], [Pi]. If X is a Shimura variety then the CM points form a countable dense set of algebraic points on X, and likewise for any Shimura subvariety Y of X. The André-Oort conjecture asserts conversely that an algebraic subvariety containing a dense set of CM points must be special, i.e., a Shimura subvariety. A concrete special case of this conjecture, proved by André, is that an irreducible algebraic curve Y in $X = \mathbb{C}^2$ containing a Zariski dense set of points whose coordinates are both j-invariants of CM elliptic curves must be either horizontal, vertical, or a modular curve $X_0(N)$.

Ghioca, Tucker, and Zhang have put forth some conjectural dynamical analogs of the Manin-Mumford conjecture [GTZ]. The main results and conjectures in the present paper can be thought of as dynamical analogs of the André-Oort conjecture. The Shimura varieties, which for our purposes can be thought of as moduli spaces for abelian varieties with certain additional structure, get replaced by moduli spaces for polynomial dynamical systems, and CM points get replaced by PCF maps. As in some approaches to the Manin-Mumford and André-Oort conjectures, equidistribution theorems for Galois orbits of special points play a crucial role in our approach to the dynamical version of these problems.

1.3. **Examples.** We now provide examples to illustrate Theorem 1.2. The first few are basic examples of families satisfying the conditions of Theorem 1.2. We include examples where the symmetries h_t are necessarily nontrivial. We conclude with two examples illustrating how Theorem 1.2 might be used to show that there are only finitely many postcritically-finite maps in a given family.

Example 1. (Infinitely many postcritically-finite maps in degree 2) In degree 2, there is a unique critical point, so the space $\mathrm{MP}_2^{cm} \simeq \mathcal{P}_2^{cm}$ is itself of dimension 1. The polynomial $f_t(z) = z^2 + t$ is postcritically finite if and only if t satisfies the polynomial equation

$$f_t^n(0) = f_t^m(0)$$

for some $n > m$. There are infinitely many such t; in fact, a simple argument involving Montel's Theorem shows that they accumulate everywhere in the boundary of the Mandelbrot set.

Example 2. (Maps with an automorphism) Let $f_t(z) = z^3 - 3t^2 z$, so $c_1(t) = t$, $c_2(t) = -t$. The orbits of c_1 and c_2 are generally disjoint, though they are symmetric by $h_t(z) = -z$. That is, we have $h_t \circ f_t = f_t \circ h_t$ and

$$f_t^n(c_1(t)) = h_t(f_t^n(c_2(t)))$$

for all t and any choice of $n \geq 0$. There are infinitely many postcritically-finite maps in this family.

Example 3. (Symmetry of the Julia set) Let $f_t(z) = z^2(z^3 - t^3)$. The Julia set of f_t has a symmetry of order 3, but f_t has no nontrivial automorphisms for $t \neq 0$. Set $\beta = (2/5)^{1/3}$ and choose $\zeta \neq 1$ so that $\zeta^3 = 1$. Then f_t has a fixed critical point at $c_1(t) = 0$ for all t, and the other critical points are $c_2(t) = \beta t$, $c_3(t) = \zeta\beta t$, $c_4(t) = \zeta^2\beta t$. Then $f_t(\zeta z) = \zeta^2 f_t(z)$ for all t, so $h(z) = \zeta z$ commutes with the second iterate f_t^2 and

$$f_t^2(c_3(t)) = \zeta f_t^2(c_2(t)), \quad f_t^2(c_4(t)) = \zeta f_t^2(c_3(t)), \quad \text{and } f_t^2(c_2(t)) = \zeta f_t^2(c_4(t))$$

for all t. There are infinitely many postcritically-finite maps in this family.

Example 4. (Symmetry h of degree > 1) Let $g_t(z) = z^2 - t^2$ and $f_t(z) = g_t^2(z) = (z^2 - t^2)^2 - t^2$ of degree 4, with $c_1(t) = 0$, $c_2(t) = t$, $c_3(t) = -t$. None of the critical points are persistently periodic, and there are infinitely many postcritically-finite parameters for the family f_t (being just the second iterate of the quadratic family). The critical points c_2 and c_3 land on c_1 after one iterate of g_t, but their orbits under f_t are disjoint from the orbit of c_1 for all $t \neq 0$; however, if we set $h(t, z) = g_t(z)$, then $f_t \circ h_t = h_t \circ f_t$ for all t, with

$$f_t(c_2(t)) = f_t(c_3(t)) \quad \text{and} \quad h_t(c_i(t)) = c_1(t)$$

for all t and $i = 2, 3$.

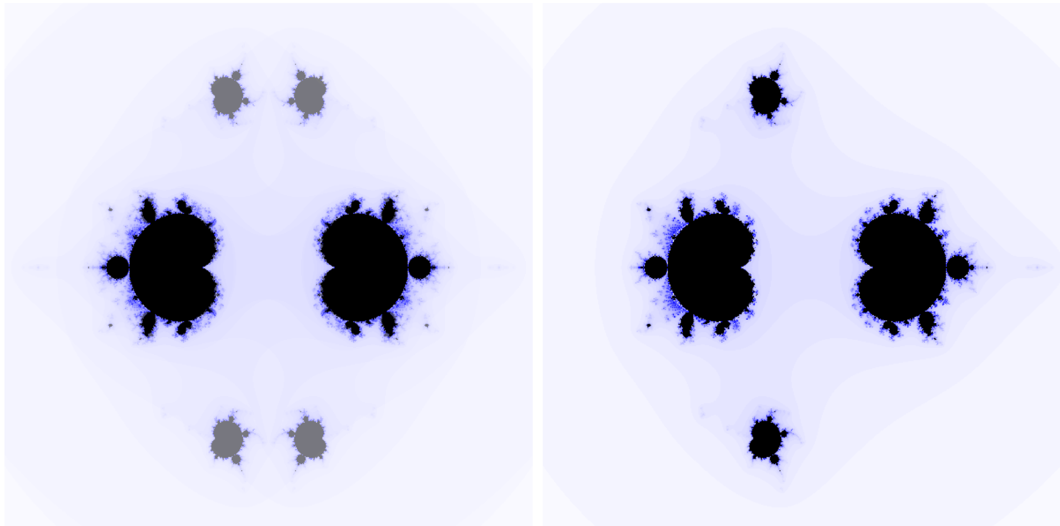

FIGURE 1.1. Left: the connectedness locus for $f_t(z) = z^3 - 3t^2 z + 0.56$ is shown in black in the region $\{-1.2 \leq \mathrm{Re}\, t \leq 1.2,\ -1.2 \leq \mathrm{Im}\, t \leq 1.2\}$; gray indicates that only one critical point remains bounded. Right: the boundedness locus M_1 for the critical point $c_1(t) = t$ is shown in black. (The boundedness locus for $c_2(t) = -t$ is the image of M_1 under $t \mapsto -t$.)

Example 5. (Finitely many PCF polynomials) In the example of Figure 1.1, there are regions in the parameter space where one critical point remains bounded while the other escapes to infinity, though the two bifurcation sets appear to have a great deal of overlap. There are only finitely many postcritically-finite maps in this family, by condition (3) of Theorem 1.2.

Example 6. (Finitely many PCF polynomials) In the family $f_t(z) = z^3 - 3t^2 + i$, we can employ condition (4) of Theorem 1.2 to show that (1) fails. Specifically, if (4) were to hold, the critical point at t would be preperiodic if and only if the critical point at $-t$ is preperiodic. So it suffices to find a single parameter t_0 at which one critical point is preperiodic while the other has infinite forward orbit. For the parameter $t_0 = i$, the critical point at $-i$ is fixed while the critical point at i lies in the basin of infinity.

1.4. **A conjecture for postcritically-finite rational maps.** Let $\{f_t : t \in V\}$ be an N-dimensional algebraic family of critically-marked rational maps of degree $d \geq 2$. In other words, V is a quasi-projective algebraic variety (over $\mathbb{C}$) of dimension N and the map $t \mapsto f_t$ defines a regular map $V \to \mathrm{Rat}_d \subset \mathbb{P}^{2d+1}_{\mathbb{C}}$ to the space of rational functions on $\mathbb{P}^1$ of degree d. Furthermore, the critical points of f_t are the images of regular maps

$$c_i : V \to \mathbb{P}^1$$

for $i = 1, \ldots, 2d-2$. Recall that the critical point c_i is active if there exists a parameter $t_0 \in V$ where $c_i(t_0)$ has infinite forward orbit for f_{t_0}. Alternatively, a family as above defines a rational function $\mathbf{f} : \mathbb{P}^1_k \to \mathbb{P}^1_k$ of degree d (where $k = \mathbb{C}(V)$ is the function field of V) with critical points $c_i \in \mathbb{P}^1(k)$, $i = 1, \ldots, 2d-2$; a critical point c_i is active if it has infinite forward orbit under $\mathbf{f}$.

If PCF maps play the role of the "special points" in the space of rational maps, then the following conjecture provides a characterization of the "special subvarieties" in the space of critically-marked rational maps Rat_d^{cm}. An n-tuple of marked critical points $(c_{i_1}, \ldots, c_{i_n})$ is said to have *dynamically dependent orbits* if there exists a relation $\{\mathbf{P} = 0\} \subset (\mathbb{P}^1_k)^n$, which is invariant under the map $(\mathbf{f}, \ldots, \mathbf{f})$, such that

$$\mathbf{P}(c_{i_1}, \ldots, c_{i_n}) = 0.$$

Invariance of X under a map F means that $F(X) \subset X$.

Conjecture 1.4. *An N-dimensional algebraic family of rational maps $\{f_t : t \in V\}$ contains a Zariski dense subset of PCF maps if and only if every $(N+1)$-tuple of active critical points has dynamically dependent orbits.*

In Theorem 1.2, our conclusion (4) is stronger than that of Conjecture 1.4 because we can appeal to the classification results of Medvedev-Scanlon [MS] to obtain a more precise form for the relation $\mathbf{P}$.

One implication of Conjecture 1.4 (dynamical dependence implies Zariski density) follows easily from an argument mimicking the proof of Proposition 2.7 and the following observation. If $N+1$ critical points have dynamically dependent orbits along V, and if N of them are preperiodic at a given parameter $t \in V$, then the $(N+1)$-th critical point will also have finite orbit at t.

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2. Activity and normal families

In this section we prove the "easy" implications in Theorems 1.1 and 1.3; see Propositions 2.5 and 2.6. The key ingredient is Montel's theory of normal families. We conclude the section with a proof that the PCF polynomials form a countable and Zariski-dense subset of $\mathcal{P}_d^{cm}$ (Proposition 2.7).

2.1. **Activity and bifurcation.** Let f_t be a holomorphic family of polynomials of degree $d \geq 2$, parameterized by $t \in \mathbb{C}$. Let $a : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Let

$$G_t(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f_t^n(z)| \tag{2.1}$$

denote the escape-rate function for f_t. Associated to any marked point is a *bifurcation measure*

$$\mu_a = \frac{1}{2\pi} \Delta G_t(a(t)), \tag{2.2}$$

where the Laplacian is with respect to t, taken in the sense of distributions.

The terminology of the bifurcation measure comes from the special case where $a(t)$ is a critical point of f_t for all t. In that case, the support of μ_a coincides with the *activity locus* of the critical point, the set of parameters where the critical point is "passing through" the Julia set of f_t. See [De1] and [DF] for background on bifurcation currents. Similarly for any marked point, the support of the measure can be characterized by a bifurcation in its dynamical properties; see e.g., [De2, Theorem 9.1].

Recall that a point $a(t)$ is active for f_t if it is not persistently preperiodic. In the special case where $a(t)$ is a critical point of f_t, the following proposition was established in [DM, Proposition 10.4] (and for rational functions in [DF, Theorem 2.5]). We give a different proof, appealing to properties of the function field height of f_t.

Proposition 2.1. *Let f_t be a family of polynomials, parameterized polynomially as*

$$f_t(z) = z^d + b_2(t) z^{d-2} + \cdots + b_d(t)$$

with $b_j(t) \in \mathbb{C}[t]$ for each j. Fix a marked point $a(t) \in \mathbb{C}[t]$. The following are equivalent:

1. *$a(t)$ is active;*
2. *$\{t \mapsto f_t^n(a(t))\}$ fails to be normal on all of $\mathbb{C}$;*
3. *$G_t(a(t)) = q \log |t| + O(1)$ as $t \to \infty$, for some positive $q \in \mathbb{Z}[1/d]$; and*
4. *the bifurcation measure*

$$\mu_a = \frac{1}{2\pi} \Delta G_t(a(t))$$

is nonzero.

Proof. We may view $\mathbf{f} = \{f_t\}$ as a polynomial defined over the function field $k = \mathbb{C}(t)$, so $\mathbf{f} \in k[z]$ and $\mathbf{a} = a(t) \in \mathbb{P}^1(k)$. If $a(t)$ is active for f_t, then $\mathbf{a}$ is not preperiodic for $\mathbf{f}$; by [Be2, Theorem B], its function-field height is positive. That is,

$$\hat{h}_{\mathbf{f}}(a) = \lim_{n\to\infty} \frac{1}{d^n} \log(\deg_t f_t^n(a(t))) > 0$$

so, in particular,

$$\deg_t f_t^n(a(t)) \to \infty$$

as $n \to \infty$ (see Remark 2.2 for more information). Choose n_0 so that $m_0 = \deg_t f_t^{n_0}(a(t)) > \max_j \deg_t b_j(t)$. Then for all $n \geq 0$,

$$\deg_t f_t^{n+n_0}(a(t)) = m_0 d^n.$$

This shows that (1) implies (3) with

$$q = \frac{m_0}{d^{n_0}}.$$

Condition (2) clearly implies condition (1). Condition (3) implies condition (4), because the function $G_t(a(t))$ cannot be harmonic on all of $\mathbb{C}$ if it has nontrivial logarithmic growth. If $\{t \mapsto f_t^n(a(t))\}$ were normal on $\mathbb{C}$, then there would be a subsequence $f_t^{n_k}(a(t))$ that converges locally uniformly in $\mathbb{C}$ to an entire function. But then the escape rate $G_t(a(t))$ would be everywhere 0. In particular, the measure μ_a would be trivial. So (4) $\implies$ (2) and the circuit of implications is closed. $\square$

Remark 2.2. We explain briefly the relation between function-field height and degree growth. Recall that if $k = \mathbb{C}(t)$ with its standard product formula structure and $\mathbf{f} \in \mathbb{C}[t, z]$ has degree d as a polynomial in $k[z]$, the canonical height $\hat{h}_{\mathbf{f}} : \mathbb{P}^1(\overline{k}) \to \mathbb{R}_{\geq 0}$ is defined for $a \in \mathbb{C}[t]$ by

$$\hat{h}_{\mathbf{f}}(a) = \lim_{n\to\infty} \frac{1}{d^n} \sum_{v \in \mathcal{M}_k} \log^+ |\mathbf{f}^n(a)|_v.$$

We can identify $\mathcal{M}_k$ with $\mathbb{C} \cup \{\infty\}$. For an absolute value v corresponding to a point $z \in \mathbb{C}$, we have $\log^+ |\mathbf{f}^n(a)|_v = 0$ since $\log |\mathbf{f}^n(a)|_v = -\operatorname{ord}_z(\mathbf{f}^n(a)) \leq 0$. For v corresponding to the point at infinity, we have $\log^+ |\mathbf{f}^n(a)|_v = \log |\mathbf{f}^n(a)|_\infty = \deg(\mathbf{f}^n(a)) \geq 0$. Thus

$$\hat{h}_{\mathbf{f}}(a) = \lim_{n\to\infty} \frac{1}{d^n} \deg(\mathbf{f}^n(a)).$$

Remark 2.3. When the conditions of Proposition 2.1 are satisfied, the measure μ_a will be compactly supported in the parameter space $\mathbb{C}$. Indeed, the function $G_t(a(t))$ is necessarily harmonic where it is positive, as it is a locally-uniform limit of harmonic functions. The set

$$\mathrm{M}_a = \{t \in \mathbb{C} : \sup_n |f_t^n(a(t))| < \infty\} = \{t \in \mathbb{C} : G_t(a(t)) = 0\}$$

will be compact. Up to a multiplicative constant (namely, the q of condition (3)), $t \mapsto G_t(a(t))$ defines the Green's function for M_a with respect to infinity, and μ_a (up to scale) is the harmonic measure of M_a with respect to infinity.

2.2. **Normality and preperiodic points.** Using Montel's theory of normal families, it is straightforward to prove that the conditions of Proposition 2.1 guarantee infinitely many parameters for which $a(t)$ has finite forward orbit. For a proof of Montel's theorem, see [Mi2, §3].

Lemma 2.4. *Suppose $a : \mathbb{D} \to \mathbb{C}$ is holomorphic and $f : \mathbb{D} \times \mathbb{C} \to \mathbb{C}$ is any holomorphic family of polynomials of degree $d \geq 2$, parameterized by the unit disk $\mathbb{D}$. Suppose $\{t \mapsto f_t^n(a(t))\}_{n\geq 1}$ fails to form a normal family on $\mathbb{D}$. Then there exists a sequence of distinct parameters $t_n \in \mathbb{D}$ for which $a(t_n)$ is preperiodic for f_{t_n} for all $n \in \mathbb{N}$. In fact, we can choose the parameter t_n so that $a(t_n)$ lands on a repelling cycle of f_{t_n} for each n.*

Proof. Let U be the largest open set in $\mathbb{D}$ on which $\{t \mapsto f_t^n(a(t))\}_{n\geq 1}$ is normal; it might be empty, and by assumption $U \neq \mathbb{D}$. Choose $t_0 \in \mathbb{D} \setminus U$, and let $\{p_1(t_0), p_2(t_0), \ldots, p_r(t_0)\}$ be any repelling cycle for f_{t_0} of period $r > 1$. By the implicit function theorem, the repelling cycle persists for t in a small neighborhood of t_0; let $p_i(t)$ denote the i-th point in the corresponding repelling cycle for f_t. Note, in particular, that $p_1(t) \neq p_2(t)$ for all t near t_0. The failure of normality on $\mathbb{D}$ and Montel's Theorem imply there exist a parameter $t_1 \in \mathbb{D}$ and an integer $k > 1$ such that

$$f_{t_1}^k(a(t_1)) \in \{p_1(t_1), p_2(t_1)\}.$$

That is, the point $a(t_1)$ is preperiodic for f_{t_1} and the cycle it lands on is repelling. Now we repeat the argument: choose any repelling cycle for f_{t_0} of period $r_2 > r$ and follow it holomorphically in a small neighborhood of t_0. We obtain a parameter t_2 so that $a(t_2)$ lands on a repelling cycle for f_{t_2}. As f_{t_0} has repelling cycles of arbitrarily high period, we may repeat the argument indefinitely. By induction, we obtain a sequence $\{t_1, t_2, t_3 \ldots\}$ of parameters where $a(t_n)$ is preperiodic for f_{t_n}, and for each n, $a(t_n)$ lands on a repelling cycle of period $r_n > r_{n-1}$. $\square$

Proposition 2.5. *Let f_t be a 1-parameter family of polynomials as in Theorem 1.3, and suppose that active points $a_1(t), a_2(t) \in \mathbb{C}[t]$ satisfy condition (3) of the theorem. Then both conditions (1) and (2) are satisfied.*

Proof. Because h_t commutes with the iterate f_t^k for all t, condition (3) implies immediately that a_1 has finite orbit at parameter t if and only if a_2 has finite orbit for f_t. Thus, condition (2) holds. For condition (1), it suffices to show that the orbit of $a_1(t)$ is finite for infinitely many parameters t. From Proposition 2.1, we know that the sequence of functions

$$\{t \mapsto f_t^n(a_1(t)) : n \geq 0\}$$

fails to form a normal family. The result then follows from Lemma 2.4. □

Proposition 2.6. *There are infinitely many postcritically-finite cubic polynomials in* $\mathrm{Per}_1(0) \subset \mathrm{MP}_3^{cm}$.

Proof. It is convenient to work in the space $\mathcal{P}_3^{cm} \simeq \mathbb{C}^2$, which is a degree-2 branched cover of MP_3^{cm}. Throughout each irreducible component of $\mathrm{Per}_1(0)$ in $\mathcal{P}_3^{cm}$, one of the marked critical points is fixed. Recalling that the connectedness locus $\mathcal{C}_3 = \{f \in \mathcal{P}_3^{cm} : J(f) \text{ is connected}\}$ is compact [BH1, Corollary 3.7], we see that both of the critical points cannot be persistently preperiodic along $\mathrm{Per}_1(0)$. Indeed, one critical point must escape to infinity (and therefore have infinite orbit) for parameters outside the connectedness locus. Thus, on each irreducible component of $\mathrm{Per}_1(0)$, exactly one critical point is active. A polynomial $f \in \mathrm{Per}_1(0)$ is postcritically-finite if the active critical point has finite forward orbit. By Lemma 2.4, there are infinitely many postcritically-finite polynomials in $\mathrm{Per}_1(0)$. □

2.3. **Countability and density of PCF maps.** To conclude this section, we provide a proof that the set of PCF maps forms a countable and Zariski dense subset of the moduli space of (critically-marked) polynomials of degree d. A sketched proof of density appears in [Si, Proposition 6.18], based on the transversality results of Adam Epstein (as appearing in [BE]), for the corresponding statement in the space of all rational functions of degree d. We provide a more direct argument for density here, from the equivalence of inactivity and normality of iterates, as first appeared in [Mc1, Lemma 2.1]. A similar proof shows that PCF maps are Zariski dense in the moduli space of rational maps. The argument that the set of PCF maps is countable (after excluding the flexible Lattès maps) requires Thurston's rigidity theorem in the case of rational maps, while we can appeal to compactness of the connectedness locus for polynomials.

Proposition 2.7. *The PCF polynomials form a countable, Zariski dense subset of* MP_d^{cm}. *The coordinates of each PCF polynomial in* $\mathcal{P}_d^{cm}$ *lie in* $\overline{\mathbb{Q}}$.

Proof. It is convenient to work in the space $\mathcal{P}_d^{cm} \simeq \mathbb{C}^{d-1}$, a branched cover of MP_d^{cm} of degree $d-1$. A postcritically-finite polynomial $f \in \mathcal{P}_d^{cm}$ is a solution to $d-1$ equations of the form

$$f^{n_i}(c_i) = f^{m_i}(c_i)$$

for integers $n_i < m_i$, $i = 1, \ldots, d-1$. As equations in the coordinates of $\mathcal{P}_d^{cm}$, they are polynomials defined over $\mathbb{Q}$. Each postcritically-finite polynomial has connected Julia set; and the connectedness locus is compact in $\mathcal{P}_d^{cm}$ [BH1, Corollary 3.7]. Consequently, the PCF maps form a countable union of algebraic sets, each contained in a compact subset of $\mathcal{P}_d^{cm}$. As any compact affine variety is finite, the collection of PCF maps is countable, and each is defined over $\overline{\mathbb{Q}}$.

We now show Zariski density. Let S be any proper algebraic subvariety of $\mathcal{P}_d^{cm}$, and let Λ be its complement. It suffices to show that there exists a PCF polynomial in Λ. The activity of a marked critical point c_i, along any quasiprojective parameter space Λ, is equivalent to the failure of normality of $\{\lambda \mapsto f_\lambda^n(c_i(\lambda))\}_{n\geq 1}$ on all of Λ; see [Mc1, Lemma 2.1] or [DF, Theorem 2.5]. Consider the critical point c_1. Either it is preperiodic along all of Λ or it is active; in either case, by applying Montel's theorem if active (as in Lemma 2.4 above), there exists a parameter $\lambda_1 \in \Lambda$ where c_1 is preperiodic. Suppose c_1 satisfies the equation $f^{n_1}(c_1) = f^{m_1}(c_1)$ at the parameter λ_1. Let $\Lambda_1 \subset \Lambda$ be the subvariety defined by this equation. Then Λ_1 is a nonempty quasiprojective variety, of codimension ≤ 1 in $\mathcal{P}_d^{cm}$, and c_1 is persistently preperiodic along Λ_1.

We continue inductively. Suppose Λ_k is a quasiprojective subvariety in $\mathcal{P}_d^{cm}$ of codimension $\leq k$, on which $c_1, \ldots, c_k$ are persistently preperiodic. If c_{k+1} is persistently preperiodic along Λ_k, set $\Lambda_{k+1} = \Lambda_k$. If not, apply Lemma 2.4 to find a parameter $\lambda_{k+1} \in \Lambda_k$ where c_{k+1} is preperiodic, and define $\Lambda_{k+1} \subset \Lambda_k$ by the critical orbit relation satisfied by c_{k+1} at λ_{k+1}. Then Λ_{k+1} has codimension at most $k+1$ in Λ, and the first $k+1$ critical points are persistently preperiodic along Λ_{k+1}. In particular, Λ_{d-1} is a nonempty subset of Λ and consists of PCF polynomials. □

Remark 2.8. An alternative proof of Zariski density in Proposition 2.7 follows from the following theorem of Dujardin and Favre: the closure of the set of postcritically-finite polynomials (in the usual analytic topology) contains the support of the bifurcation measure in MP_d^{cm} [DF, Corollary 6]. The bifurcation measure μ_{bif} cannot charge pluripolar sets [DF, Proposition 6.11], and so the PCF maps are Zariski dense.

3. Arithmetic equidistribution

In this section we recall a general arithmetic equidistribution theorem which will be used in the sequel. We state this result in a form which is a hybrid of the terminology from [BR] and [FRL]; the proof follows directly from the arguments in either of those works.[2] The result is most naturally formulated using Berkovich spaces; see [BR] for an overview.

Let k be a *product formula field*. This means that k is equipped with a set $\mathcal{M}_k$ of pairwise inequivalent nontrivial absolute values, together with a positive integer N_v for each $v \in \mathcal{M}_k$, such that:

(PF1) For each $\alpha \in k^\times$, we have $|\alpha|_v = 1$ for all but finitely many $v \in \mathcal{M}_k$.

(PF2) Every $\alpha \in k^\times$ satisfies the *product formula*

$$\prod_{v\in\mathcal{M}_k} |\alpha|_v^{N_v} \ = \ 1 \ .$$

[2] A closely related equidistribution theorem was proved independently by Chambert-Loir [CL].

Examples of product formula fields are number fields and function fields of normal projective varieties.

Let $\overline{k}$ (resp. k^{sep}) denote a fixed algebraic (resp. separable) closure of k. For $v \in \mathcal{M}_k$, let k_v be the completion of k at v, let $\overline{k}_v$ be an algebraic closure of k_v, and let $\mathbb{C}_v$ denote the completion of $\overline{k}_v$. For each $v \in \mathcal{M}_k$, we fix an embedding of $\overline{k}$ in $\mathbb{C}_v$ extending the canonical embedding of k in k_v. For each $v \in \mathcal{M}_k$, we let $\mathbb{P}^1_{\mathrm{Berk},v}$ denote the Berkovich projective line over $\mathbb{C}_v$, which is a canonically defined path-connected compact Hausdorff space containing $\mathbb{P}^1(\mathbb{C}_v)$ as a dense subspace. If v is Archimedean, then $\mathbb{C}_v \cong \mathbb{C}$ and $\mathbb{P}^1_{\mathrm{Berk},v} = \mathbb{P}^1(\mathbb{C})$.

For each $v \in \mathcal{M}_k$ there is a naturally defined distribution-valued Laplacian operator Δ on $\mathbb{P}^1_{\mathrm{Berk},v}$. For example, the function $\log^+|z|_v$ on $\mathbb{P}^1(\mathbb{C}_v)$ extends naturally to a continuous real valued function $\mathbb{P}^1_{\mathrm{Berk},v}\backslash\{\infty\} \to \mathbb{R}$ and

$$\Delta \log^+ |z|_v = \delta_\infty - \lambda_v,$$

where λ_v is the uniform probability measure on the complex unit circle $\{|z| = 1\}$ when v is archimedean and λ_v is a point mass at the Gauss point of $\mathbb{P}^1_{\mathrm{Berk},v}$ when v is non-archimedean.

A probability measure μ_v on $\mathbb{P}^1_{\mathrm{Berk},v}$ is said to have *continuous potentials* if $\mu_v - \lambda_v = \Delta g$ with $g : \mathbb{P}^1_{\mathrm{Berk},v} \to \mathbb{R}$ continuous. If μ has continuous potentials then there is a corresponding *Arakelov-Green function* $g_\mu : \mathbb{P}^1_{\mathrm{Berk},v} \times \mathbb{P}^1_{\mathrm{Berk},v} \to \mathbb{R} \cup \{+\infty\}$ which is characterized by the differential equation $\Delta_x g_\mu(x, y) = \delta_y - \mu$ and the normalization $\iint g_\mu(x, y)d\mu(x)d\mu(y) = 0$. The function g_μ is finite-valued and continuous outside of

$$\mathrm{Diag_v} := \{(z, z) \in \mathbb{C}_v \times \mathbb{C}_v\} \subseteq \mathbb{P}^1_{\mathrm{Berk},v} \times \mathbb{P}^1_{\mathrm{Berk},v}.$$

If ρ, ρ' are measures on $\mathbb{P}^1_{\mathrm{Berk},v}$ and $\mu = \mu_v$ is a probability measure with continuous potentials, we define the *μ-energy* of ρ and ρ' by

$$(\rho, \rho')_\mu := \frac{1}{2} \iint_{\mathbb{P}^1_{\mathrm{Berk},v}\times\mathbb{P}^1_{\mathrm{Berk},v}\backslash\mathrm{Diag}} g_\mu(x, y)d\rho(x)d\rho(y).$$

One can show that if ρ and ρ' have total mass zero then $((\rho, \rho')) := (\rho, \rho')_\mu$ is independent of μ; in this case our definition and notation coincide with those of Favre and Rivera-Letelier [FRL].

An *adelic measure* on $\mathbb{P}^1$ (with respect to the product formula field k) is a collection $\mu = \{\mu_v\}_{v\in M_k}$ of probability measures on $\mathbb{P}^1_{\mathrm{Berk},v}$, one for each $v \in \mathcal{M}_k$, such that:

(AM1) $\mu_v = \lambda_v$ for all but finitely many $v \in \mathcal{M}_k$.
(AM2) μ_v has continuous potentials for all $v \in \mathcal{M}_k$.

For a finite subset S of $\mathbb{P}^1(k^{\mathrm{sep}})$ and $v \in \mathcal{M}_k$, we denote by $[S]_v$ the discrete probability measure on $\mathbb{P}^1_{\mathrm{Berk},v}$ supported equally on all elements of the $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-orbit of S. The *canonical height* of S with respect to the adelic measure μ is defined

to be

$$\hat{h}_\mu(S) := \sum_{v \in \mathcal{M}_k} N_v \cdot ([S]_v, [S]_v)_{\mu_v}.$$

(For a justification of the term 'canonical height', see for example [BR, Lemma 10.27].) This is a Weil height function, in the sense that there is a constant C such that $|h(z) - \hat{h}_\mu(z)| \le C$ for all $z \in k^{\mathrm{sep}}$, where h is the standard logarithmic height on $\mathbb{P}^1$.

Theorem 3.1. [BR, FRL] *Let S_n be a sequence of pairwise disjoint finite subsets of $\mathbb{P}^1(k^{\mathrm{sep}})$. Assume that $\#S_n \to \infty$ and that $\hat{h}_\mu(S_n) \to 0$ as $n \to \infty$. Then $[S_n]_v$ converges weakly to μ_v as $n \to \infty$ for all $v \in \mathcal{M}_k$.*

Remark 3.2. When k is a number field, Theorem 3.1 is essentially the same as Theorem 2 of [FRL]. Special cases of Theorem 3.1, for arbitrary k, are proved in Theorems 7.52 and 10.24 of [BR]. It is straightforward to prove the general case of Theorem 3.1 (for arbitrary k) by using [BR, Lemma 7.55] in conjunction with the proof of [FRL, Theorem 2], as in the proof of [BR, Theorem 7.52].

Remark 3.3. If k is a number field and S_n is the set of $\mathrm{Gal}(\overline{k}/k)$-conjugates of z_n, then $\#S_n \to \infty$ follows automatically from the assumption that $\hat{h}_\mu(S_n) \to 0$ by Northcott's theorem and the fact that h_μ is a Weil height.

Remark 3.4. Rather than assuming $\hat{h}_\mu(S_n) \to 0$ as $n \to \infty$, it is enough to make the *a priori* weaker assumption that $\limsup_{n\to\infty} \hat{h}_\mu(S_n) \le 0$. This is implicit in [BR] and is proved explicitly in [FRL, Theorem 1], which asserts that $\liminf_{n\to\infty} \hat{h}_\mu(S_n) \ge 0$.

In order to apply Theorem 3.1 in practice, one usually needs to know how to explicitly compute the Arakelov-Green's functions $g_{\mu_v}(x, y)$ for $v \in \mathcal{M}_k$. There is a particularly nice way to do this when each μ_v is the equilibrium measure of a compact set $E_v \subset \mathbb{A}^1_{\mathrm{Berk},v}$, which will always be the case for the applications in the present paper. In order to explain how this works, we introduce some terminology.

Fix a place v of k and suppose that μ_v is the equilibrium measure for a compact set $E_v \subset \mathbb{A}^1_{\mathrm{Berk},v}$. Let $G_v : \mathbb{A}^1_{\mathrm{Berk},v} \to \mathbb{R}$ be the Green's function for E_v, which by assumption is continuous (i.e., we assume that E_v is a *regular* set). Let γ_v be the Robin constant of E_v, so the logarithmic capacity of E_v is $e^{-\gamma_v}$ and $G_v(s) = \log|s|_v + \gamma_v + o(1)$ as $s \to \infty$.

Define $H_v : \mathbb{C}_v^2 \to \mathbb{R}$ by

$$H_v(s,t) = \begin{cases} G_v(s/t) + \log|t|_v & t \neq 0 \\ \log|s|_v + \gamma_v & t = 0. \end{cases}$$

Then H_v is continuous and scales logarithmically, i.e., $H_v(\alpha s, \alpha t) = H_v(s,t) + \log|\alpha|_v$.

The following formula comes from a straightforward calculation which we omit.

Proposition 3.5. *The normalized Arakelov-Green function $g_{\mu_v}(x,y)$ with respect to μ_v is given, for $x,y \in \mathbb{P}^1(\mathbb{C}_v)$, by the explicit formula*

$$g_{\mu_v}(x,y) = -\log|\tilde{x} \wedge \tilde{y}|_v + H_v(\tilde{x}) + H_v(\tilde{y}) - \gamma_v, \tag{3.1}$$

where $\tilde{x}, \tilde{y}$ are arbitrary lifts of x, y to $\mathbb{C}_v^2 \setminus \{0\}$ and $(s_1,t_1) \wedge (s_2,t_2) = s_1 t_2 - s_2 t_1$.

Remark 3.6. For v archimedean, the fact that $g_{\mu_v}(x,y)$ is normalized implies (and in fact is equivalent to) the statement that $e^{-\gamma_v}$ is the homogeneous capacity (in the sense of [De2]) of the set $K = \{(s,t) \in \mathbb{C}^2 : H \le 0\}$. This is proved in a slightly more roundabout way in [De2, §4].

Applying the product formula to (3.1), we obtain:

Corollary 3.7. *Let $\mu = \{\mu_v\}_{v \in \mathcal{M}_k}$ be an adelic measure such that μ_v is the equilibrium measure associated to a compact set $E_v \subset \mathbb{A}^1_{\mathrm{Berk},v}$ for all $v \in \mathcal{M}_k$. Assume that the global Robin constant $\gamma := \sum N_v \gamma_v$ is zero. Let $S \subset k$ be a $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-stable finite set such that $G_v(z) = 0$ for every $v \in \mathcal{M}_k$ and every $z \in S$. Then $\hat{h}_\mu(S) = 0$.*

4. Cubic polynomials and fixed point multipliers

Our goal in this section is to prove Theorem 1.1. One implication was already shown in Proposition 2.6. It remains to show that for each $\lambda \neq 0$, there are only finitely many conformal conjugacy classes of postcritically-finite polynomials with a fixed point of multiplier λ. We apply the arithmetic equidistribution results described in §3.

Though our proof does not use this, we remark that it suffices to study the curves for $|\lambda| > 1$. Indeed, if $0 < |\lambda| \le 1$, there are no postcritically-finite maps on $\mathrm{Per}_1(\lambda)$; see e.g., [Mi2, Corollary 14.5].

4.1. Parameterization of $\mathrm{Per}_1(\lambda)$.

Fix $\lambda \in \mathbb{C} \setminus \{0\}$. To study the curve $\mathrm{Per}_1(\lambda)$ in the moduli space of cubic polynomials with marked critical points, it is convenient to work with the following parameterization:

$$f_s(z) = \lambda\, z - \frac{\lambda}{2}\left(s + \frac{1}{s}\right) z^2 + \frac{\lambda}{3}\, z^3$$

for $s \in \mathbb{C} \setminus \{0\}$. The polynomial f_s has a fixed point at $z = 0$ with multiplier λ and critical points at $c_+(s) = s$ and $c_-(s) = 1/s$. It is conjugate to the centered polynomial

$$P_s(z) = \frac{\lambda}{3}\, z^3 + \left(\frac{\lambda}{2} - \frac{\lambda}{4}\left(s^2 + \frac{1}{s^2}\right)\right) z + \frac{1}{12}\left(s + \frac{1}{s}\right)\left(6 - 4\lambda + \lambda s^2 + \frac{\lambda}{s^2}\right)$$

with critical points at $\pm(s^2-1)/(2s)$. Therefore, the family f_s projects to the curve $\mathrm{Per}_1(\lambda)$ within $\mathcal{P}_3^{cm}$ via

$$s \mapsto \left(\sqrt{\frac{\lambda}{3}}\,\frac{s^2-1}{2s}, -\sqrt{\frac{\lambda}{3}}\,\frac{s^2-1}{2s}, -\frac{1}{12}\sqrt{\frac{\lambda}{3}}\left(s+\frac{1}{s}\right)\left(6-4\lambda+\lambda s^2+\frac{\lambda}{s^2}\right)\right)$$

for either choice of $\sqrt{\lambda}$. This projection is generically one-to-one. This curve in $\mathcal{P}_3^{cm}$ then projects to $\mathrm{Per}_1(\lambda)$ in MP_3^{cm} with degree two, via the identification of $(c_1(s), c_2(s), b(s))$ with $(-c_1(s), -c_2(s), -b(s)) = (c_2(s), c_1(s), -b(s))$.

4.2. **The bifurcation measures.** Consider the escape-rate functions

$$G^+(s) = \lim_{n\to\infty} \frac{1}{3^n} \log^+ |f_s^n(s)|$$

and

$$G^-(s) = \lim_{n\to\infty} \frac{1}{3^n} \log^+ |f_s^n(1/s)|.$$

An induction argument shows immediately that $f_s^n(s)$ is a polynomial in s for all n. In fact,

$$f_s^n(s) = \frac{\lambda}{3}\left(\frac{\lambda}{3}\right)^3 \cdots \left(\frac{\lambda}{3}\right)^{3^{n-2}} \left(\frac{-\lambda}{6}\right)^{3^{n-1}} s^{3^n} + O(s^{3^n-1}) \in \mathbb{C}[s],$$

so

$$G^+(s) = \log|s| + \log|\lambda/3|^{1/6} + \log|\lambda/6|^{1/3} + o(1) \tag{4.1}$$

as $s \to \infty$ and $G^+(s)$ is bounded for s near 0. By symmetry, $G^-(s) = G^+(1/s)$, so G^- has a logarithmic singularity at $s = 0$ and remains bounded as $s \to \infty$.

Lemma 4.1. *For each $\lambda \neq 0$, both critical points of f_s are active.*

Proof. This follows immediately from the nontrivial growth of G^+ and G^-. □

The bifurcation measures of the critical points $c_+(s) = s$ and $c_-(s) = 1/s$ are defined by

$$\mu_+ = \frac{1}{2\pi}\Delta G^+$$

and

$$\mu_- = \frac{1}{2\pi}\Delta G^-$$

on $\mathbb{C}\setminus\{0\}$. From the growth of G^+ and G^-, we see that μ_+ and μ_- define probability measures on $\hat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$. The support of μ_+ is compactly contained in $\mathbb{C}$, and it does not put positive mass on $s = 0$. Similarly for μ_-.

The bifurcation locus for the family $\{f_s\}$ is the set of parameters s_0 where the Julia sets $J(f_s)$ fail to vary continuously (in the Hausdorff topology) on any neighborhood of s_0. The bifurcation locus coincides with $(\operatorname{supp}\mu_+)\cup(\operatorname{supp}\mu_-)$; see [De1, Theorem 1.1] for a proof.

We thank Curt McMullen for suggesting the proof of this next lemma:

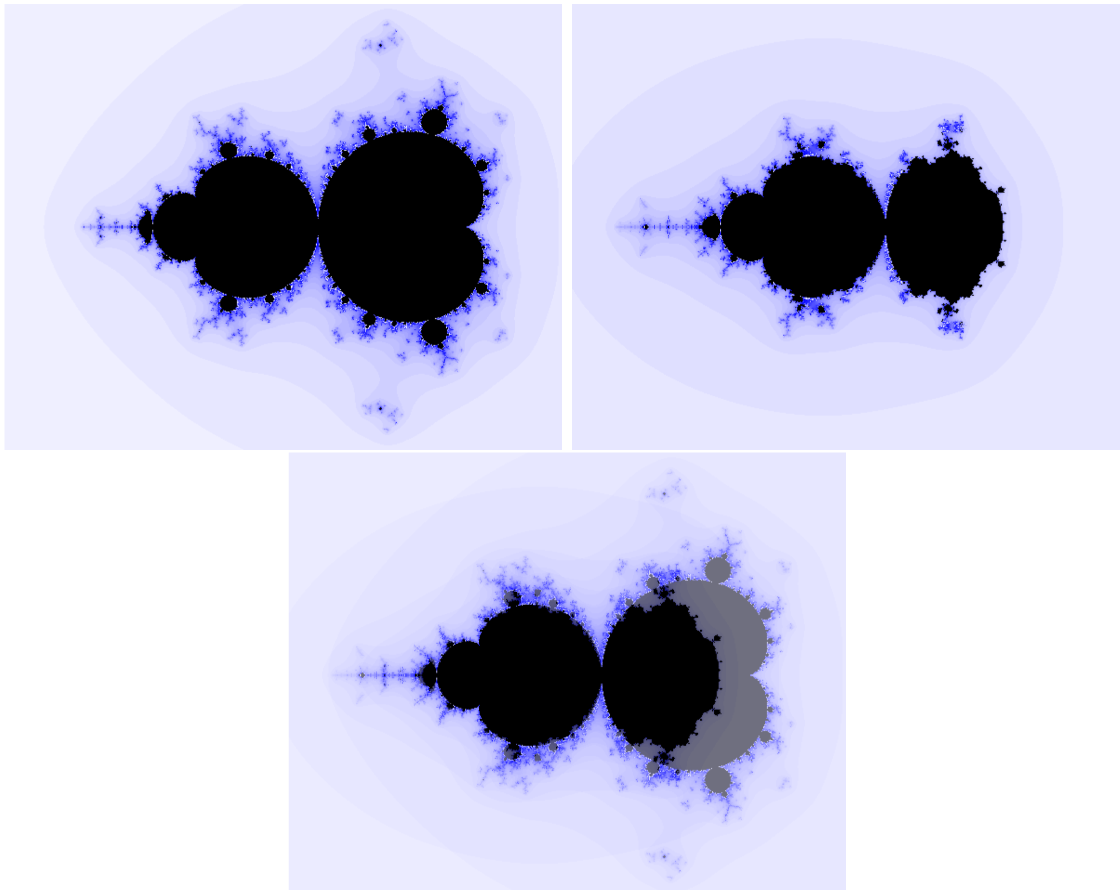

FIGURE 4.1. Top left: the support of μ_+ in $\mathrm{Per}_1(6)$, in the region $\{-1.8 \leq \mathrm{Re}\, s \leq -1.3,\ -0.2 \leq \mathrm{Im}\, s \leq 0.2\}$, with the level sets of G^+. Top right: the support of μ_- and level sets of G^- in the same region. Bottom: the two images are superimposed, and the connectedness locus appears in black. The polynomial f_s is PCF for $s = -(1+\sqrt{5})/2$, where the two critical points form a cycle of period 2.

Lemma 4.2. *The bifurcation measures μ_+ and μ_- are not equal in* $\mathrm{Per}_1(\lambda)$.

Proof. Suppose $\mu_+ = \mu_-$. Let B denote the bifurcation locus, so $B = \mathrm{supp}\, \mu_+ = \mathrm{supp}\, \mu_-$ is compactly contained in $\mathbb{C} \setminus \{0\}$. The function $G^+ - G^-$ must be harmonic on $\mathbb{C} \setminus \{0\}$, and from the computation of the escape-rate functions above, $G^+ - G^-$ grows logarithmically at each end. Therefore, $G^+(s) - G^-(s) = C + \log |s|$ for some constant C. Therefore $B = \{G^+ = G^- = 0\} \subset \{G^+ - G^- = 0\}$, so B is a subset of a circle. But the bifurcation locus B must contain homeomorphic copies of the Mandelbrot set, by the universality of ∂M [Mc2]. This is a contradiction. $\square$

4.3. **Proof of Theorem 1.1.** For $\lambda = 0$, Proposition 2.6 states that there are infinitely many postcritically-finite polynomials in $\mathrm{Per}_1(0)$.

Now suppose $\lambda \neq 0$. Suppose there are infinitely many postcritically-finite polynomials in the family f_s. Because all PCF polynomials are defined over $\overline{\mathbb{Q}}$ (Proposition 2.7), the multiplier λ is algebraic, and therefore the family f_s is defined over a number field k.

We now set up the technical apparatus needed to apply arithmetic equidistribution (Theorem 3.1). We use homogeneous coordinates on both the parameter space and the dynamical space. For each place v of k, let $\mathbb{C}_v$ be the completion of an algebraic closure of the completion of k with respect to v, and define

$$F_{(s,t)} : \mathbb{C}_v^2 \to \mathbb{C}_v^2$$

by

$$F_{(s,t)}(z,w) = \left(\lambda\, zw^2 - \frac{\lambda}{2}\left(\frac{s}{t} + \frac{t}{s}\right) z^2 w + \frac{\lambda}{3}\, z^3\,,\ w^3\right)$$

with $(s,t) \in \mathbb{C}_v^* \times \mathbb{C}_v^*$. Note that $F_{(s,t)} = F_{(t,s)}$ and $f_s(z)$ is the first coordinate of $F_{(s,1)}(z,1)$. We define

$$H_v^+(s,t) = \lim_{n\to\infty} \frac{1}{3^n} \log \|F_{(s,t)}^n(s,t)\|_v$$

and

$$H_v^-(s,t) = \lim_{n\to\infty} \frac{1}{3^n} \log \|F_{(s,t)}^n(t,s)\|_v,$$

where $\|(a,b)\|_v = \log\max(|a|_v, |b|_v)$. Both H_v^+ and H_v^- satisfy

$$H_v^\pm(\alpha s, \alpha t) = H_v^\pm(s,t) + \log|\alpha|_v$$

for any $\alpha \in \mathbb{C}_v^*$.

Note that

$$G_v^+(s) = H_v^+(s,1) = \log|s|_v + \log|\lambda/3|_v^{1/6} + \log|\lambda/6|_v^{1/3} + o(1) \tag{4.2}$$

as $s \to \infty$ by the same calculation as in (4.1), and that $G_v^+(s)$ extends continuously to $\mathbb{A}^1_{\mathrm{Berk},v}$. Moreover, one sees easily that:

(G1) $G_v^+(s)$ is the Green's function relative to ∞ for the set

$$E_v^+ = \{z \in \mathbb{A}^1_{\mathrm{Berk},v} \ :\ G_v^+(z) = 0\}.$$

In particular, the Robin constant for E_v^+ is $\gamma_v = \log|\lambda/3|_v^{1/6} + \log|\lambda/6|_v^{1/3}$ by (4.2) and the global Robin constant $\gamma = \sum N_v\gamma_v$ is equal to zero by the product formula.

(G2) $G_v^+(s) = 0$ whenever the polynomial f_s is PCF.

Let μ_v^+ be the equilibrium measure for E_v^+ (when v is archimedean, this coincides with the probability measure μ_+ introduced in §4.2) and let $\mu^+ = \{\mu_v^+\}_{v\in\mathcal{M}_k}$ be the corresponding adelic measure. (Note that this is indeed an adelic measure, as it is straightforward to verify that E_v^+ is the unit disk $\{z \in \mathbb{A}^1_{\mathrm{Berk},v} \ :\ |z|_v \le 1\}$ in $\mathbb{A}^1_{\mathrm{Berk},v}$ for all but finitely many places v of k.) Let $s_n \in \overline{k}$, $n \in \mathbb{N}$, denote an infinite sequence

of parameters such that f_{s_n} is PCF, and let S_n be the set of $\mathrm{Gal}(\overline{k}/k)$-conjugates of s_n. By (G1) and (G2), the hypotheses of Corollary 3.7 are satisfied and we conclude that $\hat{h}(S_n) = 0$ for all n. By the arithmetic equidistribution theorem (Theorem 3.1, see also Remark 3.3), it follows that $[S_n]_v$ converges weakly to μ_v^+ for all $v \in \mathcal{M}_k$.

The same considerations apply to G_v^- and μ_v^- so the same reasoning shows that $[S_n]_v$ converges weakly to μ_v^- for all $v \in \mathcal{M}_k$ as well. Consequently, we deduce that $\mu_v^+ = \mu_v^-$ for all places v of k. In particular, letting v be an archimedean place of k, we have $\mu_+ = \mu_-$, contradicting Lemma 4.2. □

5. From coincidence to an algebraic relation

In this section, we complete the proof of Theorem 1.3. The implications (3) $\implies$ (2) $\implies$ (1) are covered by Proposition 2.5. Throughout this section, we assume condition (1). We combine the arithmetic equidistribution theorem (Theorem 3.1) with techniques from complex analysis to obtain (3).

5.1. **Preliminary definitions.** Let G_t denote the escape-rate function for f_t, as defined in (2.1), and set

$$G_i(t) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f_t(a_i(t))| = G_t(a_i(t)). \tag{5.1}$$

Define the bifurcation measure

$$\mu_i = \frac{1}{2\pi}\Delta G_i \tag{5.2}$$

on the parameter space; by Proposition 2.1, the activity of a_i implies that the measure is nonzero. In fact, we see from the proof of Proposition 2.1 that the total mass of μ_i can be computed by the degree growth of the polynomials $f_t^n(a_i(t))$ as $n \to \infty$. If we pass to a high enough iterate $f_t^{N_i}(a_i(t))$, then

$$\deg_t f_t^{N_i+n}(a_i(t)) = m_i d^n$$

for some integer $m_i > 0$ and all $n \geq 0$. Then

$$G_i(t) = \frac{m_i}{d^{N_i}} \log|t| + O(1)$$

as $t \to \infty$; consequently, the measure μ_i has total mass m_i/d^{N_i}.

For the remainder of the proof, it will be convenient to replace a_i with its iterate $f_t^{N_i}(a_i(t))$. We may therefore assume that

$$\deg_t(f_t^n(a_i(t))) = m_i d^n \tag{5.3}$$

for all $n \geq 0$ and

$$\int_{\mathbb{C}} \mu_i \ = \ m_i\,. \tag{5.4}$$

5.2. **(1) $\implies$ "almost (2)" via arithmetic equidistribution.** By assumption, there are infinitely many parameters $t_1, t_2, \ldots \in \mathbb{C}$ such that both $a_1(t_n)$ and $a_2(t_n)$ are preperiodic for f_{t_n}. Following the arguments in [GHT1], Theorem 3.1 implies that the sets

$$S_1 = \{t : a_1(t) \text{ is preperiodic for } f_t\}$$

and

$$S_2 = \{t : a_2(t) \text{ is preperiodic for } f_t\}$$

differ by at most finitely many elements. If we know that the family f_t and marked points a_i are defined over $\overline{\mathbb{Q}}$, then equidistribution guarantees that $S_1 = S_2$.

We explain how this follows from [GHT1]. We have already replaced each a_i by a suitably large iterate so that condition (5.1) from [GHT1] and the conclusion of their Lemma 5.2 are satisfied for $i = 1, 2$. Then for any product formula field k over which f_t and $a_i(t)$ are defined, Corollary 6.11 of [GHT1] guarantees that the hypotheses of the equidistribution result Theorem 3.1 are satisfied. Consequently, we have $\mu_{1,v} = \mu_{2,v}$ for all places v of k. Here $\mu_{i,v}$ denotes the equilibrium measure on the set $\mathrm{M}_{i,v}$, the closure in $\mathbb{A}^1_{\mathrm{Berk},v}$ of the set of $t \in \mathbb{C}_v$ for which $a_i(t)$ is bounded under iteration of f_t. It follows that the associated canonical heights $\hat{h}_1$ and $\hat{h}_2$ must be equal. If k is a number field, the desired equality $S_1 = S_2$ follows, because $S_i = \{t \in \bar{k} : \hat{h}_i(t) = 0\}$ in this case. The general case follows from Proposition 10.5 of [GHT1]. Note that the hypothesis (i) in Theorem 2.3 of [GHT1] is not needed for any of these conclusions.

5.3. **The boundedness locus** M**.** Consider the "generalized Mandelbrot set" associated to a_i, defined by

$$\mathrm{M}_i := \{t \in \mathbb{C} : \text{the orbit of } a_i(t) \text{ is bounded}\}.$$

As with the usual Mandelbrot set, the boundary of M_i coincides with the set of parameters where $\{t \mapsto f_t^n(a_i(t))\}$ fails to form a normal family in any neighborhood. The set S_i, where a_i is preperiodic, is a subset of M_i. From Lemma 2.4, the closure of S_i contains the boundary of M_i. And, exactly as for the usual Mandelbrot set, the Maximum Principle guarantees that the complement of M_i is connected. Thus, the conclusion of §5.2 (that S_1 and S_2 differ in at most finitely many elements) guarantees that $\mathrm{M}_1 = \mathrm{M}_2$. We let M denote this common set, so

$$\mathrm{M} := \mathrm{M}_1 = \mathrm{M}_2$$

is the *boundedness locus* for a_1 and a_2. From Remark 2.3, the set M is compact.

Recall that the function G_i defined in (5.1) is, up to a multiplicative constant, the Green function for M_i (see Remark 2.3; cf. [GHT1, Lemma 6.10]). It follows that

$$G_2(t) = \alpha\, G_1(t) \quad \text{and} \quad \mu_2 = \alpha\, \mu_1$$

where

$$\alpha = \frac{m_2}{m_1} = \frac{\deg a_2(t)}{\deg a_1(t)} \tag{5.5}$$

by equation (5.4).

We will also need the "uniformizing coordinate" φ_M associated to the compact set $\mathrm{M} \subset \mathbb{C}$. This is the uniquely determined univalent function defined in a neighborhood of infinity, with $\varphi_M(t) = t + O(1)$ near ∞, such that

$$\log|\varphi_M(t)| = \frac{1}{m_i}\, G_i(t)$$

for $i = 1, 2$. It exists because the periods of the conjugate differential

$$d^* G_i = -\frac{\partial G_i}{\partial y}\, dx + \frac{\partial G_i}{\partial x}\, dy$$

lie in $2\pi m_i \mathbb{Z}$ (for loops around infinity with t large); see, e.g., [Ah, Chapter 4, §6.1].

5.4. **Analytic relation between a_1 and a_2.** Let φ_t denote the uniformizing Böttcher coordinate for f_t. That is, for each fixed t, φ_t is defined and univalent in a neighborhood of infinity and is uniquely determined by the conditions that $\varphi_t(f_t(z)) = \varphi_t(z)^d$ and $\varphi_t(z) = z + O(1)$ for all t. The Böttcher coordinate satisfies

$$\log|\varphi_t(z)| = G_t(z)$$

where it is defined. See e.g., [Mi2, §9].

The following Lemma appears as [GHT1, Proposition 7.6], but we include a proof for completeness. The arguments are similar to our proof of [BD, Lemma 3.2].

Lemma 5.1. *For each $i = 1, 2$, there exists an integer n_i so that the iterate $f_t^{n_i}(a_i(t))$ lies in the domain of φ_t for all sufficiently large t.*

Proof. For each t, let $M(f_t)$ denote the maximal critical escape rate, so

$$M(f_t) = \max\{G_t(c) : f_t'(c) = 0\}.$$

The natural domain of φ_t is

$$\{z \in \mathbb{C} : G_t(z) > M(f_t)\}.$$

The polynomial growth of the coefficients of f_t implies that $M(f_t)$ grows logarithmically in t. Indeed, by passing to a finite cover of the punctured disk $\{|t| > R\}$ for some $R >> 0$, we may assume that the critical points of f_t are holomorphic functions of t. Applying [DM, Proposition 10.4], which uses standard distortion estimates for univalent functions, we conclude that

$$M(f_t) = e \log|t| + O(1)$$

as $t \to \infty$ for some $e > 0$.

From Proposition 2.1, for each i we know that

$$G_t(a_i(t)) = q_i \log|t| + O(1)$$

for some $q_i > 0$ as $t \to \infty$. Choosing n_i so that

$$q_i d^{n_i} > e,$$

we conclude that $f_t^{n_i}(a(t))$ lies in the domain of φ_t for all t sufficiently large. $\square$

For the rest of this section, we will replace $a_i(t)$ with its iterate $f_t^{n_i}(a_i(t))$ from Lemma 5.1.

Write each polynomial as

$$a_i(t) = \zeta_i t^{m_i} + o(t^{m_i})$$

for some nonzero $\zeta_i \in \mathbb{C}$ and t near infinity. Define

$$\Phi_i(t) = \varphi_t(a_i(t))$$

so that

$$\Phi_i(t) = \zeta_i t^{m_i} + o(t^{m_i})$$

for t near infinity. Set

$$L = \operatorname{lcm}\{m_1, m_2\}$$

and write

$$L = k_1 m_1 = k_2 m_2.$$

Now Φ_1, Φ_2 are analytic maps near infinity, and each satisfies

$$|\Phi_i(t)| = \exp(G_t(a_i(t)))$$

so that

$$\Phi_i(t) = \zeta_i \, \varphi_M(t)^{m_i}$$

for each i, where φ_M is the uniformizing coordinate for M, defined in §5.3; this is because an analytic map is uniquely determined by its absolute value, up to a rotation. Therefore,

$$\varphi_t(a_2(t))^{k_2} = \frac{\zeta_2^{k_2}}{\zeta_1^{k_1}} \, \varphi_t(a_1(t))^{k_1}.$$

Set $\zeta = \zeta_2^{k_2}/\zeta_1^{k_1}$. Then for every n, we have

$$\varphi_t(f_t^n(a_2(t)))^{k_2} = \varphi_t(a_2(t))^{k_2 d^n} = (\zeta \varphi_t(a_1(t))^{k_1})^{d^n} = \zeta^{d^n} \varphi_t(f_t^n(a_1(t)))^{k_1}. \tag{5.6}$$

We will refer to (5.6) as the *analytic relation* between the orbits of $a_1(t)$ and $a_2(t)$.

Lemma 5.2. *The ζ of the analytic relation (5.6) satisfies $|\zeta| = 1$.*

Proof. Recall that the constant α from (5.5) is given by $\alpha = m_2/m_1$ and the integers k_1 and k_2 were chosen so that $k_1 m_1 = k_2 m_2$. Consequently,

$$\log|\Phi_2(t)| = G_t(a_2(t)) = \alpha\, G_t(a_1(t)) = \frac{k_1}{k_2} G_t(a_1(t)) = \frac{k_1}{k_2}\log|\Phi_2(t)|$$

and so

$$|\zeta_2|^{k_2}|\varphi_M(t)|^{m_2k_2} = |\Phi_2(t)|^{k_2} = |\Phi_1(t)|^{k_1} = |\zeta_1|^{k_1}|\varphi_M(t)|^{m_1k_1}.$$

We see that

$$|\zeta| = |\zeta_2|^{k_2}/|\zeta_1|^{k_1} = 1,$$

so that

$$|\varphi_t(a_2(t))|^{k_2} = |\varphi_t(a_1(t))|^{k_1}$$

from (5.6). □

5.5. **Properties of the Böttcher coordinate.** Our next goal will be to promote the analytic relation (5.6) to an algebraic relation between the orbits of a_1 and a_2. To achieve this, we need some estimates on φ_t. Write the expansion of φ_t for z near ∞ as

$$\varphi_t(z) = z + \sum_{s=1}^{\infty} g_s(t) z^{-s}.$$

The constant term is 0 because all f_t are centered. Note that $\varphi_t(z)$ is analytic in both t and z, where defined.

Lemma 5.3. *The coefficient $g_s(t)$ is polynomial in t for all s.*

Proof. Recall that

$$\varphi_t(f_t^n(z)) = (\varphi_t(z))^{d^n},$$

for any n. Expand both sides as series in z, so we have

$$f_t^n(z) + O(z^{-d^n}) =$$

$$z^{d^n} + cg_1(t)z^{d^n-2} + cg_2(t)z^{d^n-3} + (cg_3(t)+c'g_1(t)^2)z^{d^n-4} + (cg_4(t)+c''g_1(t)g_2(t))z^{d^n-5} + \cdots$$

for nonzero constants $c, c', c'', \ldots$ depending only on d and n.

As the coefficients of the principal part of the left-hand side are polynomials, an induction argument allows us to conclude that the $g_s(t)$ is polynomial for every s. □

Let $m = \min\{m_1, m_2\}$, where m_i is the degree in t of $a_i(t)$.

Lemma 5.4. *The degree of $g_s(t)$ in t is no greater than $m(s+1)$.*

Proof. For fixed t, choose $R = R_t$ minimal such that $\{z : |z| > R\}$ lies in the domain of the univalent function φ_t. Then

$$f(z) = R/\varphi_t(R/z)$$

defines a univalent function on the unit disk, with $f(0) = 0$ and $f'(0) = 1$. Expand f in a power series around 0, so

$$f(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Littlewood's Theorem implies that $|b_n| \leq e\,n$ for all n and any such f; see [Du, §2.4]. In fact, by the Bieberbach Conjecture (Theorem of de Branges), we know that $|b_n| \leq n$, but this is not necessary for us.

In our case, the first few terms in the expansion of f are

$$f(z) = z - g_1 R^{-2} z^3 + g_2 R^{-3} z^4 + (g_1^2 - g_3) R^{-4} z^5 + (2g_1 g_2 - g_4) R^{-5} z^6 + \cdots$$

An induction argument implies that

$$|g_s| \leq C_s R^{s+1}$$

for some constant depending on s, where g_s and R both depend on t.

Now, recall that $a_i(t)$ lies in the domain of φ_t for all t sufficiently large, by Lemma 5.1 (and the comment following the proof). From the traditional distortion arguments applied to φ_t (applying the estimate $|b_2| \leq 2$; see [BD, Lemma 3.2] or [BH1, §3]), the region

$$\{z : G_t(z) > d\, M(f_t)\}$$

lies outside the disk of radius R_t for all t large. And so we may assume that $a_i(t)$ lies outside the disk of radius R_t for all t large. That is, we have $R_t \leq |a_i(t)|$ for $|t| >> 0$, and we conclude that

$$|g_s(t)| \leq C_s |a_i(t)|^{s+1}$$

for $i = 1, 2$. Finally, then, the degree of g_s must be no greater than the degree of $a_i(t)^{s+1}$. □

5.6. **Polynomial relation between $a_1(t)$ and $a_2(t)$.** Expand each power of the Böttcher coordinate $\varphi_t(z)$ in Laurent series near infinity as

$$(\varphi_t(z))^k = P_t^k(z) + \sum_{s=1}^{\infty} b_s^k(t) z^{-s}.$$

By Lemma 5.3, the expression $P_t^k(z)$ is polynomial in both t and z; in z it is monic and centered of degree k. By Lemma 5.4, we may conclude that

$$\deg_t b_s^k(t) \leq m(s+k). \tag{5.7}$$

Indeed, b_s^k is a sum of products $\prod_{i=1}^{l} g_{s_i}$ for some $l \in \{1, \ldots, k\}$ where $\sum s_i = k - l + s$, and each product has degree at most $\sum_{i=1}^{l} m(s_i + 1) = ml + m \sum s_i = ml + mk - ml + ms = m(s+k)$.

Setting $z = f_t^n(a_i(t))$, we have

$$(\varphi_t(f_t^n(a_i(t))))^k = P_t^k(f_t^n(a_i(t))) + \sum_{s=1}^{\infty} b_s^k(t)(f_t^n(a_i(t)))^{-s}.$$

By (5.7), the infinite-sum term is $O(t^{-m(d^n-k-1)})$. Recall from equation (5.6) we have

$$\varphi_t(f_t^n(a_2(t)))^{k_2} = \zeta^{d^n}\varphi_t(f_t^n(a_1(t)))^{k_1}$$

for all $n \geq 0$. Expanding both sides in t implies that the polynomial parts of both sides must be equal for any n. Thus, for all $n >> 0$, we have

$$P_t^{k_2}(f_t^n(a_2(t))) = \zeta^{d^n} P_t^{k_1}(f_t^n(a_1(t))). \tag{5.8}$$

It will be convenient to replace a_1 and a_2 with higher iterates so that equation (5.8) holds for *all* n.

We would like to know that the polynomial relation (5.8) between $f_t^n(a_2(t))$ and $f_t^n(a_1(t))$ is independent of n; or at least that the constants ζ^{d^n} cycle through only finitely many values. We thank Dragos Ghioca for suggesting the strategy for this proof of Lemma 5.5.

Lemma 5.5. *The ratio $\zeta = \zeta_2^{k_2}/\zeta_1^{k_1}$ is a root of unity.*

Proof. We combine the analytic relation (5.6) and the polynomial relation (5.8) to obtain

$$\sum_{s=1}^{\infty} b_s^{k_2}(t)(f_t^n(a_2(t)))^{-s} = \zeta^{d^n} \sum_{s=1}^{\infty} b_s^{k_1}(t)(f_t^n(a_1(t)))^{-s} \tag{5.9}$$

for all n. Let s_1 be the smallest s for which $b_s^{k_1}$ is nonzero and s_2 the smallest s for which $b_s^{k_2}$ is nonzero. Let $C(s,k)$ denote the degree of the polynomial $b_s^k(t)$; recall from (5.7) that $C(s,k) \leq s(m+k)$. Expanding both sides of (5.9) in t, the leading term on the left-hand-side is

$$c_2\zeta_2^{-s_2d^n} t^{C(s_2,k_2)-s_2m_2d^n}$$

for some constant $c_2 \in \mathbb{C}^*$, while the leading term on the right-hand-side is

$$c_1\zeta^{d^n}\zeta_1^{-s_1d^n} t^{C(s_1,k_1)-s_1m_1d^n}$$

for some constant $c_1 \in \mathbb{C}^*$. As we have equality in (5.9) for all n, it follows that $s_2m_2 = s_1m_1$. As $L = k_1m_1 = k_2m_2$ is the least common multiple of m_1 and m_2, we may write $s_1 = \ell k_1$ and $s_2 = \ell k_2$ for some positive integer ℓ. Furthermore, the coefficients of the leading terms must coincide, so

$$\frac{c_2}{c_1} = \zeta^{d^n}\left(\frac{\zeta_2^{k_2}}{\zeta_1^{k_1}}\right)^{\ell d^n} = \zeta^{d^n+\ell d^n}$$

for all n. Therefore, ζ is a root of unity. □

Remark 5.6. The proof of Lemma 5.5 is elementary but somewhat unenlightening. When the points $a_i(t)$ are critical (i.e., in the setting of Theorem 1.2), one can give a more conceptual proof that ζ is a root of unity, as follows. From Lemma 5.2, we know that $|\zeta| = 1$. From (5.6), the argument of ζ is equal to the difference in argument between $\varphi_t(a_2(t))^{k_2}$ and $\varphi_t(a_1(t))^{k_1}$, independent of t. We are assuming that there are infinitely many parameters t such that f_t is PCF, and all periodic cycles for a PCF map must be superattracting or repelling [Mi2, Corollary 14.5]. From §5.2 and Lemma 2.4, there are infinitely many $t \in \partial\mathrm{M}$ such that both $a_1(t)$ and $a_2(t)$ are preperiodic to repelling cycles. Such a parameter t_0 will be a landing point of a "rational external ray" for φ_{M} (see e.g. [Mi2, Chapter 18]). In other words, the points $a_1(t_0)$ and $a_2(t_0)$ will be landing points for rational external rays in the Julia set of f_{t_0}. It follows that the difference in argument between $\varphi_t(a_1(t))$ and $\varphi_t(a_2(t))$ is rational, and therefore that ζ is a root of unity.

Lemma 5.5 implies that the sequence $\{\zeta^{d^n} : n \geq 0\}$ will eventually cycle. Replacing a_1 and a_2 with iterates will allow us to assume that ζ itself is periodic for z^d. That is, we may assume there exists a positive integer k so that

$$\zeta^{d^k} = \zeta.$$

Equation (5.8) can be formulated as

$$P_t^{k_2}(f_t^{kn}(a_2(t))) = \zeta\, P_t^{k_1}(f_t^{kn}(a_1(t))) \tag{5.10}$$

for all n.

5.7. **Simplifying the algebraic relation (5.10) and concluding the proof.** Define polynomials

$$A_t(z) := P_t^{k_1}(z) \qquad B_t(z) := \zeta P_t^{k_2}(z).$$

Then (5.10) implies that the algebraic curve (or a subset of the irreducible components of this curve)

$$\{(x, y) : A_t(x) = B_t(y)\} \subset \mathbb{P}^1 \times \mathbb{P}^1$$

is invariant for the map

$$(f_t^k, f_t^k) : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$$

for every t.

If the polynomial $A_t(x) - B_t(y)$ is reducible for all t, let $Q_t(x, y)$ denote a factor such that $Q_t(a_1(t), a_2(t)) = 0$ and Q_t is irreducible for general t. There are only finitely many irreducible components, so by passing to higher iterates (of the a_i and of the f^k preserving ζ), we may assume that the curve

$$C_t = \{Q_t(x, y) = 0\}$$

is invariant for (f_t^k, f_t^k) for all t, and C_t is irreducible for general t.

We now appeal to the classification of (f, f)-invariant curves in $\mathbb{P}^1 \times \mathbb{P}^1$ for polynomials f. It was treated in great generality by Medvedev and Scanlon in [MS], applying Ritt's study of polynomial decompositions from [Ri1]. As the family f_t is nontrivial, the polynomial f_t^k cannot be conjugate to z^{d^k} or a Chebyshev polynomial for all t. Therefore, the curve C_t must be a graph, of the form $\{y = h_t(x)\}$ or $\{x = h_t(y)\}$, for a polynomial h_t that commutes with f_t^k [MS, Theorem 6.24].

In other words, there exists a polynomial $h \in \mathbb{C}[t, z]$ such that $h_t \circ f_t^k = f_t^k \circ h_t$ for all t and so that either $a_2(t) = h_t(a_1(t))$ or $a_1(t) = h_t(a_2(t))$ for all t. (Recall that we have repeatedly replaced the original a_i with an iterate $f_t^{n_i}(a_i(t))$.) If the conclusion is that $a_1(t) = h_t(a_2(t))$, then the proof of condition (3) is complete. Suppose instead that $a_2(t) = h_t(a_1(t))$. If $\deg_z h = 1$, then we may replace h_t with h_t^{-1} to achieve the conclusion of condition (3). If $\deg_z h_t > 1$, then from Ritt's work we know that h_t must share an iterate with f_t; say $h_t^q = f_t^r$ [Ri1, Ri2]. Then $h_t^{q-1}(a_1(t)) = f_t^r(a_2(t))$, so we again achieve the conclusion of condition (3), taking the new h to be $h_t^{q-1}(z)$. This concludes the proof.

6. Proof of Theorem 1.2

In this final section, we provide the proof of Theorem 1.2. In most respects, Theorem 1.2 is a special case of Theorem 1.3.

(1) $\implies$ (2). Let $a_1(t)$ and $a_2(t)$ denote any pair of active critical points of f_t. At each postcritically-finite polynomial f_t, both $a_1(t)$ and $a_2(t)$ have finite forward orbit. From Theorem 1.3, condition (1) implies that the sets S_1 and S_2 coincide. In addition, as observed in §5.3, the sets M_1 and M_2 must coincide, and therefore so do their harmonic measures (relative to ∞). From Remark 2.3, the harmonic measure on M_i is exactly the bifurcation measure for the critical point a_i, normalized to have total mass 1.

(2) $\implies$ (3). For each active critical point c_i, the support of the bifurcation measure μ_i is equal to the (outer) boundary of the set M_i. Each M_i is full (meaning that its complement is connected): indeed, on a bounded component of $\mathbb{C} \setminus \mathrm{M}_i$, the Maximum Principle guarantees that the magnitude of $f_t^n(c_i(t))$ never exceeds its maximum value on M_i. Therefore the measure μ_i determines M_i as a set. And so M_i does not depend on the choice of active critical point. In particular, all critical points have bounded forward orbit for f_t if and only if $t \in \mathrm{M}_i$ for some active critical point i. Therefore, M_i is the connectedness locus for f_t.

(3) $\implies$ (4). This implication is exactly as in the proof of Theorem 1.3. Specifically, the arguments of §5.3–5.7 start with the assumption that the sets M_i coincide and conclude with the desired algebraic relation (4).

(4) $\implies$ (1). If a critical point is not active, then by definition it is preperiodic for all parameters t. If there is only one active critical point, then Proposition 2.1 and Lemma 2.4 imply that it has finite orbit for infinitely many t, and therefore f_t is postcritically finite for infinitely many t. If there are at least two active critical points, then Theorem 1.3 guarantees that they are simultaneously preperiodic at infinitely many parameters t. Again we conclude that f_t is postcritically finite for infinitely many t.

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